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Telegrapher's equation for light derived from the transport equation

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Abstract

Shortcomings of diffusion theory when applied to turbid media such as biological tissue makes the development of more accurate equations desirable. Several authors developed telegrapher's equations in the well known P_1 approximation. The method used in this paper is different: it is based on the asymptotic evaluation of the solutions of the equation of radiative transport with respect to place and time for all values of the albedo. Various coefficients for the telegrapher's equations were derived, restricted to the case of isotropic scattering, and their properties are discussed. A correct diffusion coefficient for the stationary case could be obtained. However, this solution did not lead to the correct phase velocity. Correct phase velocities in combination with a correct diffusion coefficient were found for a dispersion relation that corresponds with anisotropic Henyey–Greenstein scattering with $g = 0.22$. It provides a time-dependent description of the fluence rate with validity for all values of the albedo.

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1. Introduction

The optical properties of biological tissues are used to obtain information about the various components contained in such tissues, and have therefore important diagnostic value. The calculation of the scattered part of the electromagnetic field interacting with the tissue requires the knowledge of the material equations, (e.g. the dielectric- and magnetic-tensor function), and in principle the full solution of Maxwell's equations. However, this leads to huge practical problems.

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A much simpler way to describe the properties of light in turbid media is provided by the so-called equation of radiative transport, which, if polarisation effects can be neglected, even reduces to a *scalar* integro-differential equation for the radiance, a quantity describing the value of the radiance in a certain direction. A very useful approximation thereof, the so-called diffusion approximation, leads to a diffusion type of equation for the intensity distribution. This approximation is widely used and very popular because of its simplicity and appeal to physical intuition [1–3].

First of all, the steady-state solution of the diffusion equation has been applied to analyze measurements with cw illumination, especially when the absorption coefficient, μ_a, is small compared to the reduced scattering coefficient, $\mu_s' = \mu_s(1-g)$. Here, μ_s is the scattering coefficient and g is the anisotropy factor, which equals the average cosine of the scattering angle. When the absorption coefficient is not small compared to the reduced scattering coefficient, the accuracy of the cw diffusion approximation decreases, as has been shown earlier [4–12].

One of the disadvantages of cw illumination in applications in biological tissue is that both scattering and absorption influence the results. Because in time-resolved measurements the separation between absorption and scattering occurs by the term $\exp(-\mu_a ct)$, this gave an impulse to develop time-resolved spectroscopy [13,14], and later frequency-domain photon-diffusion [15,16] which have been applied to biological tissue with the aim to obtain the absorption and scattering coefficients of the tissue separately, and also to improve the possibilities for imaging in biological tissue [17].

Models based on diffusion theory have been applied to these time-resolved techniques [18–20].

However, it was noted that for a consistent description of time-dependent diffusion theory, the term $\exp(-\mu_a ct)$ had to describe the absorption effect and the diffusion coefficient had thus to be written as $D = 1/\mu_s$ [21,22]. Application of this latter approach led to discrepancies, because the results with this diffusion coefficient for the steady-state solution calculated with this approach are not in agreement with the more rigorous solution of the equation of transport [6–12]. The apparent diffusion coefficient of the diffusion equation, which is in agreement with the rigorous solution of the transport equation, depends on the absorption coefficient as well [10,11,23]. Furthermore, it depends on the phase function, which describes the angular distribution of the scatterers [9]. It was concluded that the problems that occur in the time-dependent diffusion equation are probably related with the basic assumption that the photon distribution over space around a point source after a peak emission is Gaussian [11]. This can only be an assumption, as this would imply that some photons travel faster than the speed of light.

A shortcoming of diffusion equations is that a local variation in photon density spreads over the medium instantaneously. In fact, the diffusion equation can become valid only after a certain time has elapsed. In contrast, a variation in photon density should initially spread with the speed of light in the medium. Furthermore, the standard diffusion theory does not take into account unscattered light. After much corrections, better results were obtained [24].

A solution for these problems may be found by examining some solutions of the telegraphers equation, TE. This equation was already examined in the 1950s [25] in a thorough discussion of the diffusion- and the telegraphers equation as approximations to the transport equation (although applied to neutron scattering). Historically, already Maxwell introduced the TE [26] but did not use the equation in his analysis of heat diffusion. In a general form with arbitrary constants α, β, γ, this equation is given by

$$\alpha \frac{\partial^2}{c^2 \partial t^2} \rho + \beta \frac{\partial}{c \partial t} \rho - \frac{1}{3} \nabla^2 \rho + \gamma \rho = 0, \tag{1}$$

which equals the time-dependent diffusion equation for the case $\alpha = 0$.

The application of the TE to light scattering in turbid media was considered [27]. It was concluded that the TE does not provide a substantial improvement of the diffusion equation. Elsewhere, it was concluded that a simple to use physical theory for these optical problems with the TE is still an unsolved problem [28,29].

It was also shown that the agreement with the rigorous solution of the transport equation could be obtained by replacing the diffusion equation by a telegraphers's equation and adjusting the speed of light [30]. However, this may not be a satisfying solution, because Eq. (1) also describes the steady-state solution, which is independent of α and β, and the disagreement that occurs with the diffusion coefficient in diffusion theory could then also be transferred to the TE.

The motivation of this work is given by the inconsistencies we have encountered also in our previous work that occurs when using the diffusion equation:

- For steady-state, the diffusion coefficient D – and thus γ in Eq. (1) which equals $\mu_a(3D)^{-1}$ in classical diffusion theory – has to be adapted in a way that depends on the absorption coefficient.
- For the time-resolved diffusion equation, the absorption-dependence of the total energy after a light pulse should only be described by the term $\exp(-\mu_a ct)$. However, this remains in conflict with the absorption-dependent diffusion coefficient.
- The diffusion equation shows an infinite velocity of the wavefront.

It is the aim of this paper to investigate the properties of the telegrapher's equation with respect to these problems. For that aim, the telegrapher's equation is derived from first principles, enabling us to determine the correct values for the various pertinent coefficients. To this end, we obtained the dispersion relations (viz. the relations between the frequencies, ω, conjugate to time, and the variables conjugate to the space variables) for the equation of radiative transport and the telegrapher's equation. These relations are then matched up to the 2nd order in frequency, which leads to the telegrapher's equation. Almost all the other derivations of the diffusion-telegrapher's equation use the so-called P_1 approximation, which essentially assumes a slightly perturbed isotropic field distribution. We refer especially to [31] for a survey and criticism of the various ways this approximation is used for the derivation of the diffusion-telegrapher's equation, and to [25] for alternative derivations.

The implication of developing around $\omega = 0$ for small values of ω is that the approximation is expected only to be valid for large values of time t, see Chapter 2 of [32], Chapter 5 of [33], Chapter 7 of [34], respectively. Similarly, if $1/v$, the Fourier variable conjugate to z, is small, the diffusion approximation is valid for large values of place z.

The remarks just made above clearly indicate the intrinsic nature of the diffusion approximation: The so-called diffusion approximation to the equation of radiative transport is describing the asymptotical behaviour of the field with respect to "large" values of time as required by the theory of Tauberian asymptotics, see Chapter 2 of [32], Chapter 5 of [33] and Chapter 7 of [34].

Because Tauberian asymptotics are commonly used for the derivation of the diffusion equation, and provide some insight into the mechanisms involved, we will stick to this method, and derive the telegrapher's equation within this approximation.

This paper is organised as follows: In Section 2, we develop the theory leading to the telegrapher's equation and derive the telegrapher's equation for isotropic scattering. The values for the coefficients of the equation are given in Sections 3 and 4 contains the discussion.

2. Theory

2.1. Derivation of the telegrapher's equation

Transport of electromagnetic radiation in absorbing and scattering media can be described by the equation of radiative transport, roughly on the condition that phase effects do not matter [35]

$$\begin{aligned}&\frac{\partial}{c\partial t}L(\vec{x},\vec{\Omega},t) + \nabla\cdot\vec{\Omega}L(\vec{x},\vec{\Omega},t)\\&\quad = -(\mu_a+\mu_s)L(\vec{x},\vec{\Omega},t)\\&\qquad + \mu_s\int \mathrm{d}^2\Omega'\, f(\vec{\Omega}'\cdot\vec{\Omega})L(\vec{x},\vec{\Omega}',t). \qquad (2)\end{aligned}$$

Here, $L = L(\vec{x},\vec{\Omega},t)$ is the radiance, the power which is transmitted through a unit area perpendicular to the direction denoted by the unit vector $\vec{\Omega}$ in position $\vec{x}$ at time t. μ_a and μ_s are the linear absorption and scattering coefficients. $f(\vec{\Omega}' \cdot \vec{\Omega})$ is the phase function, which gives the probability that a photon incident from the direction Ω is scattered into direction Ω'. The phase function is normalized so that $\int d^2\Omega' f(\vec{\Omega}' \cdot \vec{\Omega}) = 1$. It is assumed to depend only on the angle between Ω and Ω'. The simplest phase function occurs when all scattering angles are equally probable. This is the case of isotropic scattering, and $f(\vec{\Omega}' \cdot \vec{\Omega}) = 1/4\pi$. In plane-symmetric geometries, the transport equation can be written [36]

$$\frac{\partial}{c\partial t}L(z,\vec{\Omega},t) + \cos\theta\frac{\partial}{\partial z}L(z,\vec{\Omega},t) + \mu_t L(z,\vec{\Omega},t) = \mu_s \int d^2\Omega' f(\vec{\Omega}' \cdot \vec{\Omega})L(z,\vec{\Omega}',t). \tag{3}$$

in which z is the coordinate orthogonal to the plane. $\mu_t = \mu_s + \mu_a$, and θ is the angle between the z-axis and $\vec{\Omega}$.

The transport equation can be solved analytically for a few simple geometries only, and is hard to solve numerically, because $L(\vec{x},\vec{\Omega},t)$ depends on six variables: three space variables, two angles, and time. This is why an approximation called the diffusion approximation or P_1-approximation is often used. The diffusion equation, viz. Eq. (1) with $\alpha = 0$, is a linear differential equation determining the fluence rate

$$\rho(\vec{x},t) \equiv \int d\vec{\Omega}' L(\vec{x},\vec{\Omega}',t). \tag{4}$$

The form of the differential equation has been determined on phenomenological grounds. We can picture the radiation in the medium as composed of photons that collide from time to time, and get absorbed or scattered.

To take into account the shortcomings of the diffusion equation that were mentioned in Section 1, in this paper the 2nd-order time derivative is added [37]

$$\alpha\frac{\partial^2}{c^2\partial t^2}\rho + \beta\frac{\partial}{c\partial t}\rho - \frac{1}{3}\nabla^2\rho + \gamma\rho = 0. \tag{5}$$

The meaning of α is that it introduces a finite propagation (phase) velocity $c(3\alpha)^{-0.5}$ [37]. The coefficient β determines the decay in time. The general case, in which α can be either nonzero (telegraphers equation) or zero (diffusion equation) will be called a general telegrapher's equation.

To obtain the coefficients of the telegrapher's equation it should be noted that Eq. (1) for $\partial\rho/\partial t = 0$ reduces to the time-independent diffusion equation. Therefore γ can be determined from the steady-state solutions.

Furthermore, we will focus on a method to determine α and β. The method which we will follow is to derive them from Eq. (2).

2.2. The coefficients of the telegrapher's equation

To obtain the coefficients of the telegrapher's equation, we can use the Ansatz [31]

$$\rho(z,t) = \rho_0 e^{-z\mu_t/v + i\omega t}. \tag{6}$$

Substitution of Eq. (6) in Eq. (1) gives

$$-\alpha\left(\frac{\omega}{c}\right)^2 + \beta i\left(\frac{\omega}{c}\right) + \gamma = \frac{\mu_t^2}{3v^2}, \tag{7}$$

i.e. the desired dispersion relation between v^{-2} and ω.

It is possible to choose α, β and γ such that this equation approximates the dispersion relation belonging to the transport equation. When this approximation is carried out in 1st order in ω, we obtain the solution of the diffusion equation, involving only β and γ. If also α is determined, the solution of the telegrapher's equation is obtained.

The corresponding ansatz in order to obtain the dispersion relation of the transport equation with propagation in the z-direction is given by

$$L(z,\vec{\Omega},t) = L_0(\vec{\Omega})e^{-z\mu_t/v + i\omega t}, \tag{8}$$

which has to be substituted into Eq. (2). The ansatz (8) is the analogue of the Fourier decomposition used for the solution of partial differential equations with constant coefficients [38]. This substitution then leads to a dispersion relation between v^{-2} and ω, which is derived below, viz. Eq. (23).

The treatment is analogous to p. 87 of [36], which comprises expanding the phase function in Legendre polynomials. Introducing the albedo

$$a = \mu_s/\mu_t, \tag{9}$$

the transport equation is

$$\frac{1}{\mu_t}\frac{\partial}{c\partial t}L(z,\vec{\Omega},t)+\cos\theta\frac{1}{\mu_t}\frac{\partial}{\partial z}L(z,\vec{\Omega},t)+L(z,\vec{\Omega},t) = a\int d^2\Omega' f(\vec{\Omega}'\cdot\vec{\Omega})L(z,\vec{\Omega}',t). \tag{10}$$

Writing the phase function as a series of Legendre polynomials,

$$f(\cos\theta)=\sum_{j=0}^{N}\frac{2j+1}{4\pi}f_jP_j(\cos\theta), \tag{11}$$

yields

$$\left(v\left(\frac{i\omega}{c\mu_t}+1\right)-\cos\theta\right)L_0(\theta) = \frac{va}{2}\sum_{j=0}^{N}(2j+1)f_jP_j(\cos\theta) \times\int_{-1}^{1}d\cos\theta'\,L_0(\theta')P_j(\cos\theta'). \tag{12}$$

Eq. (12) can be written in a form similar to the solution of Eq. (12) for $\omega = 0$, by new introduction of the variables $\tilde{a}$ and $\tilde{v}$ defined by

$$v\to\tilde{v}=v\left(1+\frac{i\omega}{c\mu_t}\right),\quad a\to\tilde{a}=a\Big/\left(1+\frac{i\omega}{c\mu_t}\right). \tag{13}$$

Eq. (12) can be written as

$$(\tilde{v}-\cos\theta)L_0(\theta)=\frac{\tilde{a}\tilde{v}}{2}\sum_{j=0}^{N}(2j+1)f_jP_j(\cos\theta) \times\int_{-1}^{1}L_0(\theta')P_j(\cos\theta')\,d\cos\theta'. \tag{14}$$

The derivation now is analogue to Case [36, p. 88]. We multiply by $P_k(\cos\theta)$ and integrate over $\cos\theta$. Abbreviating

$$\Phi_k\equiv\int_{-1}^{1}d\cos\theta'\,P_k(\cos\theta')L_0(\theta') \tag{15}$$

and using the orthogonality of Legendre polynomials

$$\int_{-1}^{1}dy\,P_k(y)P_j(y)=\frac{2}{2k+1}\delta_{jk}, \tag{16}$$

together with

$$yP_j(y)=\frac{1}{2j+1}[(j+1)P_{j+1}(y)+jP_{j-1}(y)], \tag{17}$$

we find that Φ_j, the Legendre coefficient of the angular distribution, is defined recursively

$$\Phi_j=\tilde{v}(1-\tilde{a}f_{j-1})\Phi_{j-1}\frac{2j-1}{j}-\frac{j-1}{j}\Phi_{j-2} \tag{18}$$

with $\Phi_0 = 1$ and $\Phi_1 = (1-\tilde{a})\tilde{v}$.

We are free to choose $\Phi_0 = 1$, because of the linearity of the transport equation. Now we can write Eq. (14) for a normalised value of $L_0(\theta)$ with $\Phi_0=\int_{-1}^{+1}L_0(\theta)\,d(\cos\theta)=1$ as

$$(\tilde{v}-\cos\theta)L_0(\theta)=\frac{1}{2}\tilde{a}\tilde{v}\sum_{j=0}^{N}(2j+1)f_j\Phi_jP_j(\cos\theta). \tag{19}$$

We divide by the first factor in the left-hand side, and integrate over θ. Since $\Phi_0 = 1$ we get

$$\Lambda(\tilde{v})\equiv 1-\frac{\tilde{a}\tilde{v}}{2}\sum_{j=0}^{N}(2j+1)f_j\Phi_j\int_{-1}^{1}d\cos\,\theta\frac{P_j(\cos\theta)}{\tilde{v}-\cos\theta} = 0. \tag{20}$$

The integral is related to the Legendre function of the second kind $Q_j(\tilde{v})$

$$Q_j(\tilde{v})=\frac{1}{2}\int_{-1}^{1}dy\,\frac{P_j(y)}{\tilde{v}-y}. \tag{21}$$

Therefore, we arrive at the following relation between $\tilde{a}$ and $\tilde{v}$, the dispersion relation:

$$\Lambda(\tilde{v})=1-\tilde{a}\tilde{v}\sum_{j=0}^{N}(2j+1)f_j\Phi_jQ_j(\tilde{v})=0. \tag{22}$$

Roots of Eq. (22) with real a and v instead of our complex $\tilde{a}$ and $\tilde{v}$ were numerically calculated by [9]. We define v to be the root with the largest absolute value, so that the mode Eq. (8) decays spatially as slowly as possible, since by that property, this mode contributes the most significant part in the diffusive regime of long distances. We would like to stress the point that the replacements of Eq. (13) and the following equations above have *general validity* for all values of the quantities involved, and will consequently be used in the calculations of Section 2.3.1.

Since Eq. (22) can be written as $\Lambda(v^{-2}, \omega, a) = 0$ this equation is an implicit equation for v^{-2} as a function of ω and an expansion parameter $q = 1 - a$. In the following, we will derive the expansion of v^{-2} in powers of ω as required by Tauberian theory [33,34]. We would like to stress the fact that in order to calculate, e.g. α the expansion (27) should be of order four, i.e. two orders higher than the 2nd order in ω! Using standard theorems on series inversion and the Lagrangian series theorem [39], Eq. (22) leads to

$$-\alpha\left(\frac{\omega}{c}\right)^2 + \beta \mathrm{i}\left(\frac{\omega}{c}\right) + \gamma = \frac{\mu_t^2}{3v^2}, \tag{23}$$

which has the same form as Eq. (7).

The complexity of Eq. (22), and hence its approximate solution equation (23), depends on the type of phase function. In the case of isotropic scattering, the equation is fairly simple and will be treated below.

2.3. Development of dispersion relations

2.3.1. Isotropic scattering

In the case of isotropic scattering, the Legendre coefficients of the phase function are all zero, except $f_0 = 1$. This relative simplicity gives us a possibility to apply the above procedure around $a = 1$, which corresponds to the case of small μ_a/μ_s. We apply this approach using a "small" parameter: $\tilde{q} = 1 - \tilde{a}$. Eq. (22) from which the series expansions of $\tilde{v}^{-2}$ into the "small" parameter are to be derived reads as

$$\Lambda(\tilde{v}) = 1 - \frac{\tilde{a}\tilde{v}}{2}\log\frac{1+\frac{1}{\tilde{v}}}{1-\frac{1}{\tilde{v}}} = 1 - \tilde{a}\tilde{v}\ \mathrm{arctanh}\frac{1}{\tilde{v}} = 0. \tag{24}$$

This is in agreement with the exact solution of v^{-2} for isotropic scattering [36]

$$\Lambda(v) = 1 - \frac{av}{2}\log\frac{1+\frac{1}{v}}{1-\frac{1}{v}} = 1 - av\ \mathrm{arctanh}\frac{1}{v} = 0. \tag{25}$$

Writing Eq. (24) as

$$1 - \tilde{a} = \frac{\tilde{a}}{3\tilde{v}^2} + \frac{\tilde{a}}{5\tilde{v}^4} + \frac{\tilde{a}}{7\tilde{v}^6} + \frac{\tilde{a}}{9\tilde{v}^8} + \cdots \tag{26}$$

shows that Eq. (24) is an implicit equation for $\tilde{v}^{-2}$ as a function of $1 - \tilde{a}$. The actual calculation of the expansion coefficients follows from the Lagrange inverse series theorem for implicit functions [39], (explicit expressions up to order 12 are to be found in [40]). Then, expressing $\tilde{v}^{-2}$ in terms of a power series expansion in powers of $\tilde{q} = 1 - \tilde{a}$, (a procedure known as "reversion of a power series") leads to

$$\frac{1}{\tilde{v}^2} = 3\tilde{q} - \frac{12}{5}\tilde{q}^2 + \frac{12}{175}\tilde{q}^3 + \frac{12}{175}\tilde{q}^4 + \cdots \tag{27}$$

2.3.2. Anisotropic scattering

A second solution of $\tilde{v}^{-2}$, that can also be used for anisotropic scattering, can be found by starting with the dispersion relation, Eq. (22), for the second stationary case between v^{-2} and $1 - a$. A fit for the rigorous solution of Eq. (22) for the stationary case was proposed by [9] for various phase functions introducing the function $F^2 = 3(1 - a)v^2$:

$$\frac{1}{F^2} = \frac{1}{3}(1 - a') + (a')^m, \tag{28}$$

with $a' = \mu_s'/(\mu_a + \mu_s')$ and appropriate values for m that showed to lead to a very accurate description of the relation between a' and v. For isotropic scattering, we may use $a' = a$. The function

$$F = \frac{\sqrt{3\mu_a\mu_t}}{\mu_t/v} = v\sqrt{3(1-a)} \tag{29}$$

is the ratio between the attenuation coefficients according to standard diffusion theory and the rigorous solution of the cw-equation of radiative transport as given in [7]. From Eqs. (28) and (29), we then derive

$$\frac{1}{v^2} = \frac{3(1-a)}{F^2} = 3(1-a)\left[\frac{1-a}{3} - a^m\right]. \tag{30}$$

For isotropic scattering, a fit can be obtained with errors smaller than 0.8% for $a > 0.15$ when choosing $m = 1.178$, [11], whereas the fit for more forwardly scattering according to the phase function of Henyey and Greenstein [41] with $g = 0.875$ fits to the exact results using $m = 0.75$ with errors <0.5% for $a' > 0.15$.

Developing Eq. (30) in a Taylor series around $a = 1$ yields

$$\frac{1}{v^2} = 3(1-a)\Big[1 + (1/3 - m)(1-a) + 0.5m(m-1)(1-a)^2 + \cdots\Big] \quad (31)$$

However, the fit of Eq. (31) is less accurate than that of Eq. (30) and gives no possibilities to adjust the fit. Therefore, Eq. (31) was written in a more general form

$$\frac{1}{v^2} = 3(1-a)\Big[1 + A(1-a) + B(1-a)^2\Big]. \quad (32)$$

The slope of the fit of the term between brackets in the vicinity of $a = 1$ is determined by A, whereas B corrects for the fit at larger distances from $a = 1$. Eq. (32) with the substitutions Eq. (13) leads to

$$\frac{1}{\tilde{v}^2} = 3(1-\tilde{a})\Big[1 + A(1-\tilde{a}) + B(1-\tilde{a})^2\Big]. \quad (33)$$

It is clear that Eq. (32) or Eq. (33) can be applied to anisotropic phase functions [9]. We applied Eq. (33) to isotropic scattering. The value for A was derived from substitution of $m = 1.178$ for isotropic scattering into Eq. (31) which gives the correct slope around $a = 1$. B was chosen such that this approximation has validity over a large range by fitting Eq. (33) to the exact value of Eq. (25) for $a = 0.3$.

2.4. Determination of the coefficients of the telegrapher's equation

Eqs. (27) and (33) as well as higher order approximations are the basic dispersion relations examined in Section 3. From this equation, we obtain Eq. (7), expanding v^{-2} as a power series of ω to the 2nd order and of $1-a$ up to the 2nd or 3rd order. With the identifications (see Eqs. (6) and (1))

$$\mathrm{i}\omega \to \frac{\partial}{\partial t}; \quad -\mu_t/v \to \frac{\partial}{\partial z}, \quad (34)$$

we then derive from Eq. (23) the coefficients α, β and γ for the various telegrapher's equations

$$\alpha\frac{\partial^2\rho}{c^2\partial t^2} + \beta\frac{\partial\rho}{c\partial t} - \frac{1}{3}\nabla^2\rho + \gamma\rho = 0. \quad (35)$$

It should be noted that the main requirement for a correct stationary solution of Eq. (35), a correct apparent diffusion coefficient [11], is only described by γ. First of all, it will be investigated in Section 3 whether this requirement is fulfilled using Eq. (27) or Eq. (33) for a given phase function.

2.5. Requirement for a correct phase velocity

A check for the goodness of the other coefficients of the telegrapher's equation that describe the time-dependent solution, α and β, is introduced here. It has been shown that the phase velocity of waves generated by Eq. (35) is independent of any specific solution of the TE but is an intrinsic property of the equation [42]. This velocity then is the quotient of the coefficients occurring in front of the operators ∇^2 and $\partial^2/\partial t^2$, ($\sqrt{1/(3\alpha)}$ in units of c), as can been seen easily for a the case of an undamped wave equation substituted into Eq. (35). Hence, the quantity $1/3\alpha$ should always be as closely as possible to 1, and thus $\alpha = 1/3$. We calculated the phase velocity generated by the telegrapher's equation in Section 3 and checked whether α is close to 1/3 for all values of the albedo.

3. Results

3.1. Stationary solution for isotropic scattering

In this section, we give the results for the coefficients α, β, and γ as calculated from Eqs. (27) and (33). The results are summarized in the Tables 1 and 2. The values for α, β, and γ in Tables 1 and 2 are obtained from the expansion of $\tilde{v}^{-2}$ into powers of ω to the 2nd order and $1-a$ to the 2nd and 3rd order.

First of all the stationary case $\omega = 0$ has been evaluated, for which α and β play no role Eq. (7) and γ is related with the exponential slope of Eq. (6). The results of γ have been compared with the results of the rigorous solution of the transport Equation, Eq. (25), using $v = \mu_t/\sqrt{\gamma}$ according to Eq. (23). The results are in Fig. 1, which shows that the 2nd- and 3rd-order Taylor approximation of the exact dispersion relation Eq. (27) lead to better results than diffusion theory, where $F = 1$ would be obtained. However, only correct results are obtained for values of a close to unity, but deviations are obtained over the whole range compared to the rigorous results and the approximation generated by Eq. (33).

Table 1
Second-order approximations for the coefficients of the telegrapher's equation, Eq. (35) using Eq. (27)

	With Eq. (27)
α	$12/175q + 12/175q^2 + 1/5$
β	$(-3/5q + 12/175q^2 + 1)\mu_t$
γ	$(q - 4/5q^2)\mu_t^2$

Higher order approximations of Eq. (27) in the mean square sense into powers of $1 - a$ were obtained for the 5th, 7th,..., 16th order which lead to even better agreement with the rigorous result from Eq. (25). Fifth order results are also shown in Fig. 1.

3.2. Check of the phase velocity requirement for isotropic scattering

The problem now arises to decide which telegrapher's equation is the "best". To this end, we observe that a common feature of the various proposed telegrapher's equations is the wrong value for the phase velocity that should equal $c/\sqrt{3\alpha}$. The derived value is in the range $1.3c$ until $1.5c$ for values of $\mu_a \cong 0$, as is easily observed from Tables 1 and 2, and hence too large. Furthermore, it depends on absorption. Because the higher order approximations have a closer fit to the rigorous results of γ, as shown in 3.1, one might expect that it might also better fit for α. Therefore, we obtained results for the phase velocity if the solution of the dispersion relation Eq. (25) is approximated in the mean square sense by powers of $1 - a$ up to the 3th, 5th, 7th, ..., 16th order into powers of $1 - a$. Fig. 2 shows the results for the 3rd-, 5th-, and 9th-order approximations. The higher approximations lead to worse results. The resulting telegrapher's equation approximation is very sensitive for the particular choice of the dispersion relation.

Table 2
Third-order approximations for the coefficients of the telegrapher's equation, Eq. (35) using Eqs. (27) and (33)

	With Eq. (27)	With Eq. (33) and $A = -0.8447, B = 0.1385$
α	$179q^3 + 17/250q^2 + 0.069q + 1/5$	$0.139q^3 – 0.4160q^2 + 0.4160q + 0.155$
β	$(0.0103q^3 + 0.07q^2 – 3/5q + 1)\mu_t$	$(-0.1420q^3 + 0.42q^2 – 0.69q + 1)\mu_t$
γ	$(0.023q^3 - 4/5q^2 + q)\mu_t^2$	$(0.134q^3 - 0.840q^2 + q)\mu_t^2$

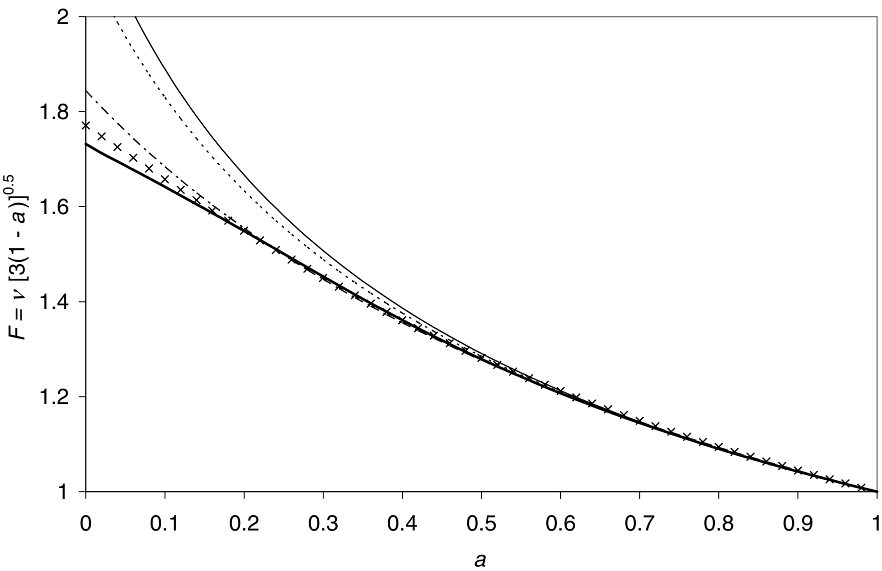

Fig. 1. Stationary solutions of the dispersion relation. Rigorous solution Eq. (25), ——; 2nd-order approximation of Eq. (27), ——; 3rd-order approximations of Eq. (27), ...; and Eq. (33) as in Table 2, -·-·-; 5th-order approximation of Eq. (27), x x x x.

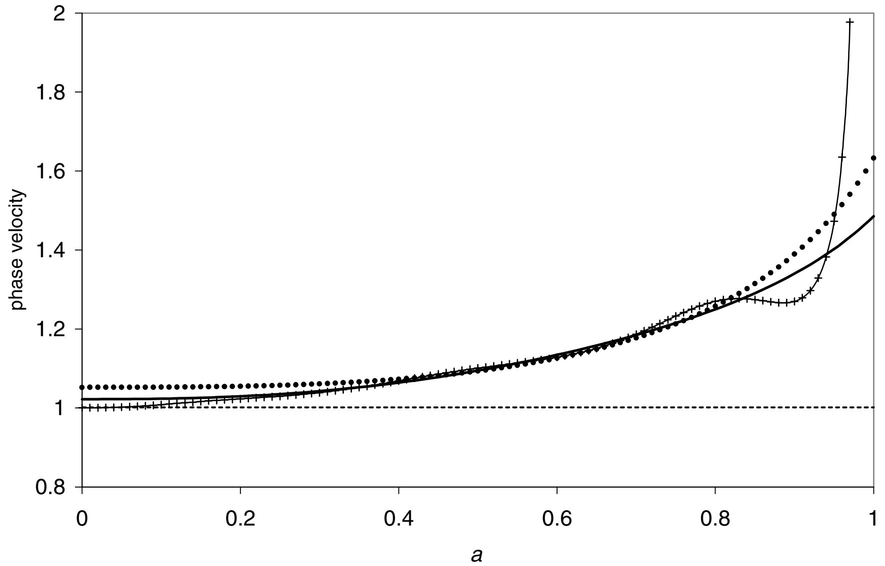

Fig. 2. Phase velocity in units c as a function of the albedo for the 3rd order, ·····; fifth order, ——; and 9th order mean square approximation of Eq. (25), +–+–+–+. The correct value of the phase velocity is 1 for all values of a.

We observe from Fig. 2 that the 5th-order approximation gives the best results: the phase velocity is for a long range of values of the albedo close to c and does not show the rapid increase of its values if the albedo is close to unity. However, these results are also incorrect.

3.3. The phase velocity requirement and values for γ

The results in Sections 3.1 and 3.2 show that none of the telegrapher's equations given above leads to correct values of both α and γ for isotropic scattering. Until now we concentrated on the "goodness" of the values of γ requiring two conditions for the calculated values: γ should as closely as possible approximate the exact values and the phase velocity $c/\sqrt{3\alpha} = c$, or $\alpha = 1/3$. An alternative approach to obtain this is now introduced. We calculated the phase velocity of the more general solution of the telegrapher's equation using Eq. (33) and require this velocity to be as closely as possible to c for all values of the albedo. Several choices for the coefficients A and B for the 3rd-order approximation Eq. (33) showed that $A = -2/3$ and $B = 0$ leads to the correct phase velocity for all values of a. The results for $A = -2/3$ and $B = 0.33$, 0.033, and 0.0033, respectively, are shown in Fig. 3.

Table 3 gives the coefficients for the telegrapher's equation for $A = -2/3$ and $B = 0.33$ and $B = 0$. We see from this table that $\alpha = 1/3$ for $B = 0$, which corresponds to the correct phase velocity.

The dispersion curves, including that with $A = -2/3$ and $B = 0$, deviate from the isotropic dispersion curve (Fig. 4). Therefore, we will consider very shortly some cases of anisotropical scattering. Fig. 4 also shows several dispersion curves (see the legend for the explanation), for various Henyey–Greenstein phase functions [41]. The values for g of the Henyey–Greenstein phase function were varied. In particular, we see close agreement between the Henyey–Greenstein phase function with $g = 0.22$. We have therefore shown the existence of a phase function leading to Telegrapher's equations with both an almost correct stationary solution and a correct phase velocity c!

4. Discussion

In the analysis given above the derivation of the telegrapher's equation is given as an approximation of the equation of radiative transport valid for sufficiently "large" values of time. Sufficiently

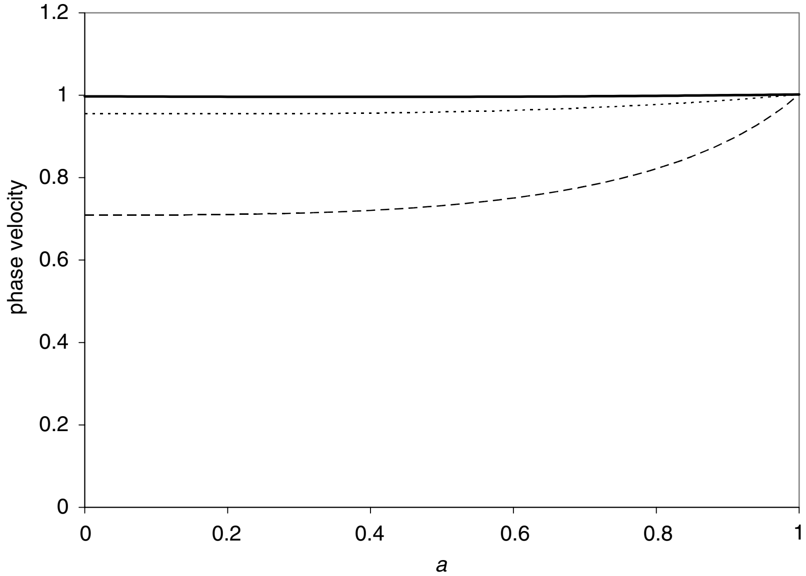

Fig. 3. Phase velocity in units c as a function of the albedo for $B = 0.33$, ——; $B = 0.033$, - - -; and $B = 0.0033$, ——.

Table 3
Third-order approximations searching for a correct phase velocity

	Eq. (33) with $A = -2/3$; $B = 0.33$	Eq. (33) with $A = -2/3$; $B = 0$
α	$0.330q^3 - 0.994q^2 + 0.994q + 0.332$	$0q^3 - 0q^2 + 0q + 0.333$
β	$(-0.340q^3 + q^2 - 0.340q + 1)\mu_t$	$(-0q^3 + 0q^2 - 0.33q + 1)\mu_t$
γ	$(0.332q^3 - 0.670q^2 + q)\mu_t^2$	$(0q^3 - 0.667q^2 + q)\mu_t^2$

"large" therefore corresponds to a distance of only more the a few $1/\mu_t$ as can be observed from Eq. (13).

In order to derive the TE, we derived the dispersion relation between ω and v^{-1}, occurring in the "Ansatz" Eq. (8). The asymptotical behaviour of the solution of the equation of radiative transport for "large" values of time and place is then obtained using Tauberian techniques [32–34], which consequently leads to the determination of a power series expansion of v^{-1} into powers of ω. We would like to stress that our method works for *all* values of the albedo a. Hence, the idea that any telegrapher's (diffusion) equation approximation to the equation of radiative transport only exists for sufficiently small values of the absorption coefficient [31] is not true. In contrast, for correct results at all values of the albedo a correct phase velocity is even required, because for the case of relative large absorption, the non-scattered photons will dominate the fluence rate at all distances from the source. The validity of our results for all values of the albedo is illustrated by the fact that substitution of $\rho(\mu_a, \vec{r}, t) = \rho_0(\mu_a = 0, \vec{r}, t)\exp(-\mu_a ct)$ into (35) shows that α, β, and γ taken from Table 3 for $B = 0$ are valid for all values of μ_a.

Our results, based on Eq. (27) lead to telegrapher's equations different from those proposed in the literature (see [31] for a survey of the proposed telegrapher's equations). Our coefficients for α, β, and γ differ from those of [30], whose telegraphers equation is not valid for $\mu_s' = 0$, where some of its coefficients become infinite. Our results also differ from the one put forward by [31], with $\alpha = 1/5$, $\beta = \mu_t(3/5a + 2/5)$, and $\gamma = \mu_t^2(1/5 + 3/5a - 4/5a^2)$. It should be noted that this solution of Eq. (1) can be obtained by our method in the following way: assume that

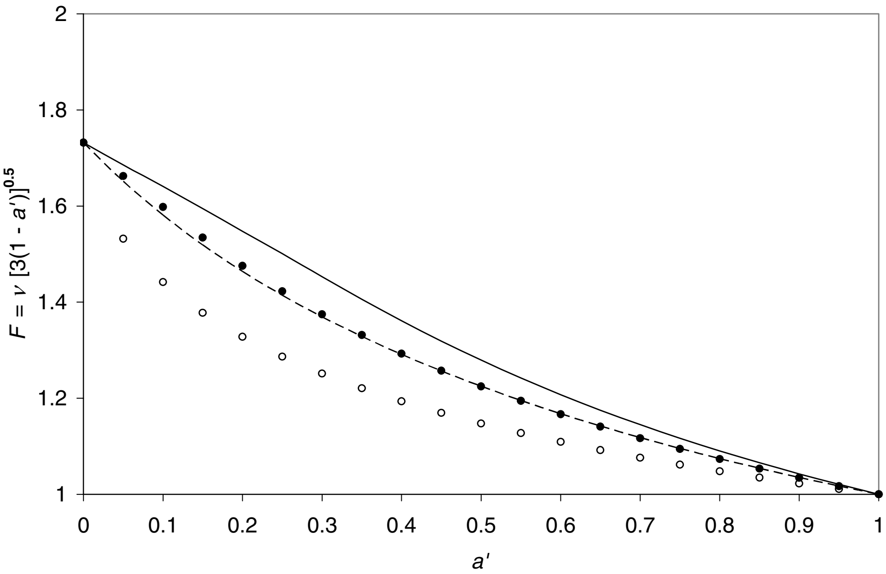

Fig. 4. The stationary solutions of the dispersion relation for correct phase velocity ($A = -2/3$, $B = 0$): ––––, comparison with rigorous solution for isotropic scattering; ——, and for anisotropic Henyey–Greenstein scattering with $g = 0.875$: ○○○, and $g = 0.22$: ●●●.

the quadratic approximation of Eq. (25), (viz. Eq. (27) without the tilde and the cubic term) is the appropriate dispersion relation. Then our method leads to Polishchuk's coefficients of the TE. However, when a 4th-order approximation of Eq. (25) is applied (viz. Eq. (27) without the tilde and including the 4th-order term), it is true that we arrive at the same 2nd-order expansion of γ as given by Polishchuk, but the values of the time-dependent coefficients α and β within this approximation are different. This indicates that a higher order approximation of Eq. (25) is needed to arrive at the correct coefficients α and β within that approximation. Unfortunately, we have shown that, using the telegrapher's equation, the higher order approximations for isotropic scattering have not led to the correct values of α.

It is clear that the telegrapher's equation preferably is used in a way that the calculated phase velocity of the light for all albedo values equals that of the phase velocity. Our results show that no correct solution could be found for isotropic scattering when also the phase velocity should equal c. A correct description for the phase velocity was only found for $F^{-2} = 1/3 + 2/3a'$. The curve in Fig. 4 that is valid for this phase function closely corresponds to an existing *anisotropical* phase function: The Henyey–Greenstein function with $g = 0.22$. This curve was found empirically.

Although from measurements described in the literature, it is expected that the phase function for biological tissue is more close to that for $g = 0.875$ than the phase function mentioned above for which Fig. 3 shows deviations up to 10%. Even under these circumstances the telegraphers equation with $A = -2/3$ and $B = 0$ gives a much better description of γ than diffusion theory, for all values of the albedo. Future studies on the use of this telegrapher's equation with the correct phase velocity probably will show this approach to be a step forward in the description of light propagation in biological tissue.

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